1. Elemente de analiză vectorială

1.2 Câmpuri scalare.

Fie o funcție scalară „f” definită într-un domeniu spațial (plan) $\mathcal{D}$ raportat la un sistem de coordonate *cartezian* (x, y, z), *cilindric* (z ,r, φ), sferic (r, θ, φ) etc. În fiecare punct din domeniu P_0 (x_0, y_0, z_0) funcția scalară are o valoare f_0 (x_0, y_0, z_0); unind punctele în care funcția f are aceeaşi valoare, se obține o *suprafață de nivel* pentru funcția f.
Prin intermediul *gradientului* se pot studia proprietățile variaționale ale funcției scalare f:

$$\mathrm{grad}\ f = \nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} \tag{1.1}$$

unde $\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$ este un operator de derivare vectorial (nabla).

Funcția „grad f” are drept componente vitezele de variație ale funcției f după coordonate ($\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$), respectiv „grad f”, în orice punct P (x, y, z) arată viteza de variație a funcției la trecerea prin acel punct ; grad f este orientat în sensul crescător al funcției f. În regiunea din $\mathcal{D}$ unde grad f este mare, acolo suprafețele de nivel (f = cst) sunt apropiate.

Variația funcției f după o direcție de versor $\bar{n}$ este:

$$\frac{\partial f}{\partial \bar{n}} = \bar{n}\ \nabla f \tag{1.2}$$

Proprietățile funcției gradient sunt:

$$\begin{cases} \mathrm{grad}(af) = a\ \mathrm{grad}\ f \\ f = m \pm n \rightarrow \nabla f = \nabla(m \pm n) = \nabla m \pm \nabla n \\ f = m \cdot n \rightarrow \nabla f = \nabla(m \cdot n) = m\nabla n + n\nabla m \\ \mathrm{grad}\ m = \mathrm{grad}\ n \rightarrow m = n + cst \\ F = F(m, n, ...) \rightarrow \mathrm{grad}\ F = \frac{\partial F}{\partial m}\nabla m + \frac{\partial F}{\partial n}\nabla n + ... \end{cases} \tag{1.3}$$

Dacă $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ este vectorul de poziție într-un sistem cartezian (figura. 1.1-a) sau $\bar{r} = z\overline{u_z} + r\overline{u_r} + r\varphi\ \overline{u_\varphi}$ într-un sistem cilindric (figura 1.1-b), iar $r = \left|\bar{r}\right| = \sqrt{x^2 + y^2 + z^2}$ este modulul vectorului de poziție, atunci:

$$\text{grad } r = \nabla r = \frac{\bar{r}}{r} \tag{1.4}$$

şi este orientat în sensul creşterii coordonatei r.

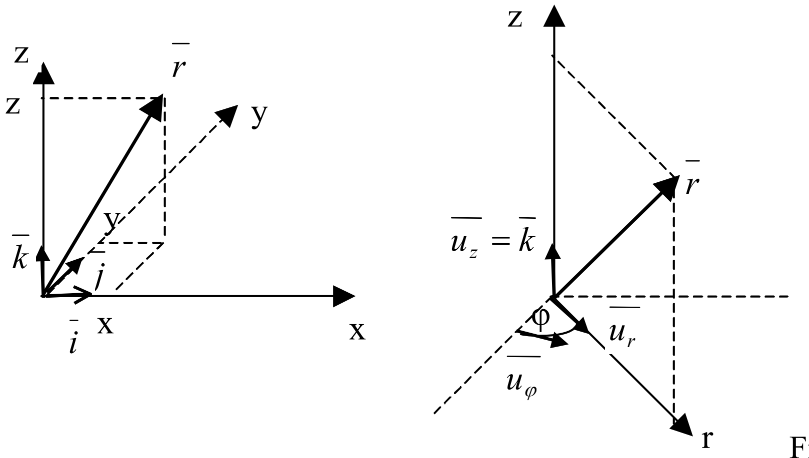


Fig. 1.1

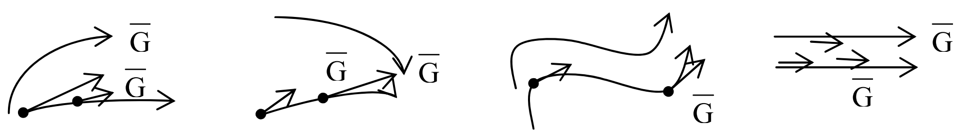
1.2 Câmpuri vectoriale.

Un câmp de vectori $\overline{G}$ definit într-un domeniu $\mathcal{D}$ este o funcție vectorială:

$$\overline{G} = \overline{G}(x, y, z) = G_x\bar{i} + G_y\bar{j} + G_z\bar{k} \tag{1.5}$$

şi în orice punct din domeniu are o *valoare* (modulul $G = \sqrt{G_x^2 + G_y^2 + G_z^2}$), o *direcție* în spațiu, un *sens* pe direcția respectivă, deci o tripletă de valori în raport cu o funcție scalară f care are doar *valoare*.

Un *tub de flux unitate* conține atâția vectori (un mănunchi de vectori) cât este unitatea de măsură a fluxului respectiv. Fiecare tub unitate se înlocuieşte prin axa sa geometrică care va reprezenta o *linie de câmp* a lui $\overline{G}$. Într-o regiune din spațiu unde câmpul este intens, tuburile unitate vor fi apropiate iar acolo unde câmpul este slab, un tub unitate se adună pe o suprafață mai mare şi liniile de câmp vor fi mai îndepărtate. Geometria liniilor de câmp sugerează multe dintre proprietățile câmpului $\overline{G}$.



a) linii divergente; b) linii convergente; c) linii echidistante; d) linii paralele
Fig.1.2

În lungul liniei de câmp valoarea câmpului scade (fig. 1.2-a), creşte (fig. 1.2-b), rămâne constantă (fig. 1.2-c) sau este un câmp uniform (fig. 1.2-d).

Câmpul $\overline{G}$ rămâne tangent la linia de câmp; vectorii $\overline{G}$ şi $\overline{dr}$ fiind coliniari: $\overline{G}\times\overline{dr}=0$, respectiv dacă $\overline{dr}=dx\overline{i}+dy\overline{j}+dz\overline{k}$, rezultă:

$$\frac{dx}{G_x}=\frac{dy}{G_y}=\frac{dz}{G_z} \tag{1.6}$$

care reprezintă ecuaţiile diferenţiale a căror soluţie sunt *ecuaţiile liniilor de câmp.*

Divergenţa unei funcţii vectoriale $\overline{G}=G_x\overline{i}+G_y\overline{j}+G_z\overline{k}$ se defineşte:

$$div\overline{G}=\nabla\overline{G}=\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z} \tag{1.7}$$

şi este o funcţie scalară care indică repartiţia surselor pentru liniile câmpului $\overline{G}$ în interiorul domeniului $\mathcal{D}$ de definiţie a câmpului $\overline{G}$.

Dacă într-un punct P_0 (x_0, y_0, z_0) din $\mathcal{D}$:

$\left(\mathrm{div}\overline{G}\right)_{P_0}>0\rightarrow$ în punctul P_0 există surse (+) care produc linii de câmp $\overline{G}$. Valoarea $\mathrm{div}\overline{G}$ arată care este productivitatea surselor din acel punct; $\left(\mathrm{div}\overline{G}\right)_{P_0}=2\rightarrow$ la trecerea prin punctul P_0 se dublează numărul liniilor de câmp

$\left(\mathrm{div}\overline{G}\right)_{P_0}<0\rightarrow$ în punctul P_0 există surse (-) numite *puţuri* care absorb linii de câmp.

$\left(\mathrm{div}\overline{G}\right)_{P_0}=0\rightarrow$ nu există surse în P_0, liniile lui $\overline{G}$ trec continuu prin acel punct (numărul liniilor nici nu creşte nici nu scade).

Proprietățile „divergenței” se pot evidenția ținând seama că operatorul nabla (∇) este un operator de derivare vectorială:

$$\begin{cases} \overline{G} = k\overline{A} \rightarrow \text{div } \overline{G} = \nabla\overline{G} = k\nabla\overline{A} = k \text{ div } \overline{A} \\ \overline{G} = f\overline{A} \rightarrow \text{div } \left(f\overline{A}\right) = \nabla\left(f\overline{A}\right) = f\nabla\overline{A} + \overline{A}\nabla f = f \text{ div } \overline{A} + \overline{A} \text{ grad } f \\ \overline{G} = \text{grad } f = \nabla f \rightarrow \text{div } \left(\text{grad } f\right) = \nabla\left(\nabla f\right) = \Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} \\ \Delta = \nabla\nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \quad - \text{operatorul laplacean} \\ \overline{G} = \overline{r} \rightarrow \text{div } \overline{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = \begin{cases} 3 \rightarrow \overline{r} \quad \text{vector spatial} \\ 2 \rightarrow \overline{r} = x\overline{i} + y\overline{j}, \quad \text{vector plan} \end{cases} \end{cases}$$

Rotorul unei funcții vectoriale $\overline{G} = G_x\overline{i} + G_y\overline{j} + G_z\overline{k}$ se definește:

$$\text{rot } \overline{G} = \nabla \times \overline{G} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix} =$$

$$= \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\overline{i} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial y}\right)\overline{j} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\overline{k} \tag{1.9}$$

deci rot $\overline{G}$ este tot o funcție vectorială care indică proprietățile geometrice ale liniilor de câmp $\overline{G}$.

rot $\overline{G}$ =0→ liniile lui $\overline{G}$ nu fac rotoare în domeniul *D*, sunt linii deschise (cu capete, început şi sfârşit).

rot $\overline{G}$ ≠0→ liniile lui $\overline{G}$ sunt închise în domeniul *D* (fac rotoare în *D*) sau trec prin domeniul *D* în fascicol paralel (deci se închid pe la infinit).

Câteva dintre proprietățile funcției „rotor” le amintim în continuare:

$$\overline{G} = \text{grad } f = \nabla f \to \text{rot }\left(\text{grad } f\right) = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} =$$

$$= \underbrace{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)}_{=0} \overline{i} + 0\overline{j} + 0\overline{k} \tag{1.10}$$

Deci dacă:

$$\text{rot } \overline{G} = 0 \quad \to \quad \overline{G} = -\text{grad } V = -\nabla V \tag{1.11}$$

funcția scalară V se numeşte *potențialul scalar* al câmpului $\overline{G}$.

Toate câmpurile $\overline{G}$ care provin dintr-un potențial scalar V prin intermediul gradientului se numesc *câmpuri potențiale* (newtoniene, irotaționale) şi liniile lor sunt deschise, nu fac rotoare.

$$\begin{aligned} &\overline{G} = f\overline{A} \to \text{rot }\left(f\overline{A}\right) = \nabla \times \left(f\overline{A}\right) = \nabla f \times \overline{A} + f\left(\nabla \times \overline{A}\right) = \\ &= \text{grad } f \times \overline{A} + f \text{ rot } \overline{A} \end{aligned} \tag{1.12}$$

Dacă $\overline{G} = \overline{A} \times \overline{B}$, atunci aplicând formula lui Gibbs se obține succesiv:

$$\begin{aligned} &\text{rot }\left(\overline{A} \times \overline{B}\right) = \nabla \times \left(\overline{A} \times \overline{B}\right) = \nabla \times \left(\underset{\bullet}{\overline{A}} \times \overline{B}\right) + \nabla \times \left(\overline{A} \times \underset{\bullet}{\overline{B}}\right) = \begin{vmatrix} \underset{\bullet}{\overline{A}} & \overline{B} \\ \nabla \underset{\bullet}{\overline{A}} & \overline{B}\nabla \end{vmatrix} + \\ &+ \begin{vmatrix} \overline{A} & \underset{\bullet}{\overline{B}} \\ \overline{A}\nabla & \nabla \underset{\bullet}{\overline{B}} \end{vmatrix} = \overline{A} \text{ div } \overline{B} - \overline{B} \text{ div } \overline{A} + \left(\overline{B} \text{ grad}\right)\overline{A} - \left(\overline{A} \text{ grad}\right)\overline{B} \end{aligned} \tag{1.13}$$

Dacă $\overline{G} = \text{rot } \overline{A} = \nabla \times \overline{A}$, aplicând (1.13) se obține:

$$\begin{aligned} &\text{rot } (\text{rot } \overline{A}) = \nabla \times \left(\nabla \times \overline{A}\right) = \\ &= \nabla\left(\nabla\ \overline{A}\right) - \underbrace{\overline{A}\left(\nabla\ \nabla\right)}_{=0} + \underbrace{\left(\overline{A}\ \nabla\right)\nabla}_{=0} - \underbrace{\left(\nabla\ \nabla\right)}_{=\Delta}\overline{A} = \text{grad }\left(\text{div } \overline{A}\right) - \Delta\ \overline{A} \end{aligned} \tag{1.14}$$

- Operatorul nabla acționează asupra cantităților din dreapta sa cu proprietăți de derivare (pe cele din stânga le înmulțeşte) iar dacă în dreapta sa nu există nimic, termenul este zero, ca în relația (1.14).

- $\overline{G} = \bar{r} \to \text{rot } \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

- Dacă $\overline{A}$ şi $\overline{B}$ sunt doi vectori, atunci se definesc produsele:

$\overline{A} \bullet \overline{B} = A_x B_x + A_y B_y + A_z B_z$ - produs scalar

$\overline{A} \times \overline{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ - produs vectorial

- produs mixt:

$$\overline{A} \bullet \left(\overline{B} \times \overline{C}\right) = \begin{cases} = 0, \text{când doi dintre vectori sunt paraleli} \\ \neq 0, \text{cei 3 vectori determină un paralelipiped} \end{cases}$$

- dublu produs vectorial (Gibbs):

$\overline{A} \times \left(\overline{B} \times \overline{C}\right) = \begin{vmatrix} \overline{B} & \overline{C} \\ \overline{AB} & \overline{AC} \end{vmatrix}$

$\text{div}\left(\text{rot}\overline{A}\right) = \nabla \bullet \left(\nabla \times \overline{A}\right) = 0$ - produs mixt în care doi vectori coincid.

Deci dacă:

$$\text{div}\overline{G} = 0 \;\; \to \;\; \overline{G} = \text{rot}\overline{A} \tag{1.15}$$

funcția vectorială $\overline{A}$ este *potențialul vector* al câmpului $\overline{G}$.

Observații

- În teoria câmpului „gradientul”, „divergența” şi „rotorul” sunt *operatori diferențiali* de ordinul întâi care exprimați prin operatorul de derivare spațială nabla: $\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\overline{k}$, se scriu pe baza proprietăților de derivare şi înmulțire simultană a acestui operator: $\text{grad } f = \nabla f$; $\text{div}\overline{G} = \nabla\overline{G}$; $\text{rot}\overline{G} = \nabla \times \overline{G}$.

- Dacă în domeniul $\mathcal{D}$, $\mathrm{rot}\overline{G}=0$, înseamnă că liniile câmpului $\overline{G}$ nu se închid în acel domeniu (dar se pot închide în exteriorul său) şi în acelaşi timp dacă şi $\mathrm{div}\overline{G}=0$, înseamnă că liniile lui $\overline{G}$ nu au surse (capete) în $\mathcal{D}$ (dar poate avea surse în exterior). Un astfel de câmp se numeşte *câmp laplacean*.

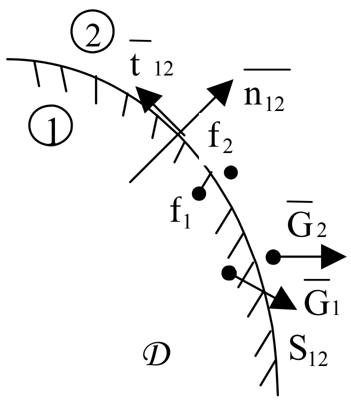


fig 1.3

Dacă în domeniul de câmp $\mathcal{D}$ există o suprafaţă de discontinuitate S_{12} care separă două regiuni cu proprietăţi materiale diferite (sau la domenii plane există o curbă de discontinuitate C_{12}) ca în figura 1.3, atunci la trecerea prin discontinuitate, în sensul versorului normal la discontinuitate n_{12} orientat dinspre mediul 1 spre 2, funcţia va avea o variaţie prin salt (f_2 – f_1), respectiv $\left(\overline{G_2}-\overline{G_1}\right)$, care reprezintă valorile celor două funcţii la stânga şi la dreapta discontinuităţii S_{12}. În acest caz nu se pot defini operatorii grad, div şi rot, dar se operează cu operatorii superficiali:

- gradient superficial: $\mathrm{grad}_s f=\overline{n_{12}}\left(f_2-f_1\right)$
- divergenţa superficială: $\mathrm{div}_s\overline{G}=\overline{n_{12}}\bullet\left(\overline{G_2}-\overline{G_1}\right)=G_{n2}-G_{n1}$
- rotor superficial: $\mathrm{rot}_s\overline{G}=\overline{n_{12}}\times\left(\overline{G_2}-\overline{G_1}\right)=G_{t2}-G_{t1}$

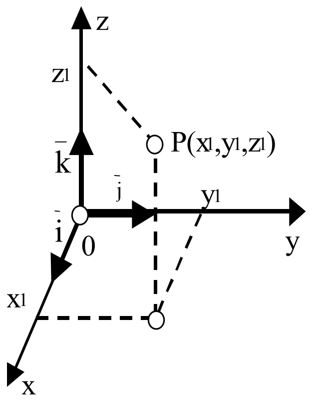
unde G_n este componenta lui $\overline{G}$ după normala la discontinuitate $\overline{n_{12}}$ iar G_t este componenta lui $\overline{G}$ după versorul tangent $\overline{t}$ la S_{12}.

Dacă $\overline{n}$ şi $\overline{t}$ sunt versorii normal şi tangent la o suprafaţă (curbă) atunci:

$$\overline{A}\bullet\overline{n}=A_n \qquad \overline{A}\times\overline{n}=\overline{A}_t$$
$$\overline{A}\bullet\overline{t}=A_t \qquad \overline{A}\times\overline{t}=\overline{A}_n$$

1.3 Proprietățile funcțiilor de punct în diverse sisteme de coordonate.

1.3.1 Sistemul cartezian (x,y,z) de versori ($\bar{i}$, $\bar{j}$, $\bar{k}$).



$$\bar{F}=F_x\bar{i}+F_y\bar{j}+F_z\bar{k}\ ;\ \ \bar{F}\,\bar{G}=F_xG_x+F_yG_y+F_zG_z;$$

$$\bar{F}\times\bar{G}=(F_yG_z-F_zG_y)\bar{i}+(F_zG_x-F_xG_z)\bar{j}+(F_xG_y-F_yG_x)\bar{k}\ ;$$

$$\text{grad }V=\frac{\partial V}{\partial x}\bar{i}+\frac{\partial V}{\partial y}\bar{j}+\frac{\partial V}{\partial z}\bar{k}\,;$$

$$\nabla=\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\,;$$

$$\text{div }\bar{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\,;$$

$$\text{rot}\,\bar{F}=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\bar{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\bar{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\bar{k}\,;$$

$$\nabla^2V=\Delta V=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}\,;\qquad \Delta\bar{F}=\text{grad div }\bar{F}-\text{rot rot }\bar{F}\,;$$

$$\nabla^2\bar{F}=\Delta\bar{F}=\nabla^2F_x\bar{i}+\nabla^2F_y\bar{j}+\nabla^2F_z\bar{k}=\Delta F_x\bar{i}+\Delta F_y\bar{j}+\Delta F_z\bar{k}\,;$$

$$\text{grad div}\,\bar{F}=\left(\frac{\partial^2F_x}{\partial x^2}+\frac{\partial^2F_y}{\partial x\partial y}+\frac{\partial^2F_z}{\partial x\partial z}\right)\bar{i}+\left(\frac{\partial^2F_x}{\partial x\partial y}+\frac{\partial^2F_y}{\partial y^2}+\frac{\partial^2F_z}{\partial y\partial z}\right)\bar{j}+$$

$$\left(\frac{\partial^2F_x}{\partial x\partial z}+\frac{\partial^2F_y}{\partial y\partial z}+\frac{\partial^2F_z}{\partial z^2}\right)\bar{k}\,;$$

$$\text{grad }\bar{k}\,\bar{F}=\bar{k}\times\text{rot}\,\bar{F}+\frac{\partial\bar{F}}{\partial z}\,;\ \text{div}(\bar{k}\times\bar{F})=-\bar{k}\,\text{rot}\,\bar{F}\,;\ \ \text{rot}(\bar{k}\times\bar{F})=\bar{k}\,\text{div}\,\bar{F}-\frac{\partial\bar{F}}{\partial z}\,;$$

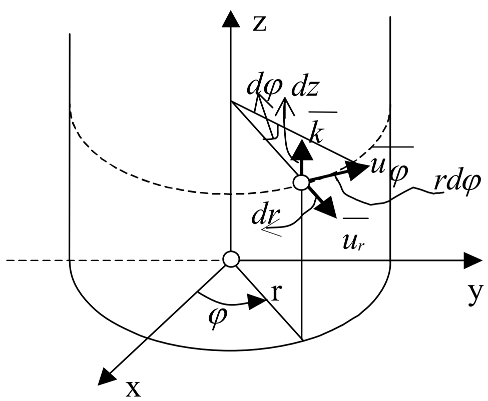
$$\text{rot rot}(\bar{k}\times\bar{F})=-\bar{k}\times\text{grad div}\,\bar{F}-\text{rot}\frac{\partial\bar{F}}{\partial z}\,;$$

$$\text{grad div }\bar{k}\,V=\frac{\partial}{\partial z}(\text{grad }V)=\text{grad}\frac{\partial V}{\partial z}\,;$$

$$\text{rotrot}\,\overline{F}=\left(-\frac{\partial^2F_x}{\partial y^2}-\frac{\partial^2F_x}{\partial z^2}+\frac{\partial^2F_y}{\partial x\partial y}+\frac{\partial^2F_z}{\partial x\partial z}\right)\overline{i}+\left(-\frac{\partial^2F_y}{\partial z^2}-\frac{\partial^2F_y}{\partial x^2}+\frac{\partial^2F_z}{\partial y\partial z}+\frac{\partial^2F_x}{\partial y\partial x}\right)\overline{j}$$
$$+\left(-\frac{\partial^2F_z}{\partial x^2}-\frac{\partial^2F_z}{\partial y^2}+\frac{\partial^2F_x}{\partial z\partial x}+\frac{\partial^2F_y}{\partial z\partial y}\right)\overline{k};$$

$$\text{rot}\,\overline{k}\,V=\text{grad }V\times\overline{k};\quad \nabla^2(\overline{k}\times\overline{F})=\overline{k}\times\nabla^2\,\overline{F};$$
$$\text{grad div}\,\overline{k}\times\overline{F}=-\overline{k}\times\text{rot rot}\,\overline{F}-\text{rot}\frac{\partial\overline{F}}{\partial z};$$
$$\nabla^2(\overline{k}\,V)=\overline{k}\;\nabla^2V;\qquad \text{rot rot}(\overline{k}\,V)=-\overline{k}\;\nabla^2V+\text{grad}\frac{\partial V}{\partial z}.$$

1.3.2 Sistemul cilindric circular (r,φ,z) de versori $(\overline{u}_r,\overline{u}_\varphi,\overline{u}_z=\overline{k})$



$$\text{rot.rot}\,\overline{F}=\left(-\frac{1}{r^2}\frac{\partial^2F_r}{\partial\varphi^2}-\frac{\partial^2F_r}{\partial z^2}+\frac{\partial^2F_z}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2F_\varphi}{\partial r\partial\varphi}+\frac{1}{r^2}\frac{\partial F_\varphi}{\partial\varphi}\right)\overline{u}_r+$$
$$\left(-\frac{\partial^2F_\varphi}{\partial z^2}+\frac{1}{r}\frac{\partial^2F_z}{\partial\varphi\partial z}-\frac{\partial^2F_\varphi}{\partial r^2}-\frac{1}{r}\frac{\partial F_\varphi}{\partial r}+\frac{F_\varphi}{r^2}-\frac{1}{r^2}\frac{\partial F_r}{\partial\varphi}+\frac{1}{r}\frac{\partial^2F_r}{\partial\varphi\partial r}\right)\overline{u}_\varphi$$
$$+\left(-\frac{\partial^2F_z}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2F_z}{\partial\varphi^2}+\frac{\partial^2F_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2F_\varphi}{\partial\varphi\partial z}+\frac{1}{r}\frac{\partial F_r}{\partial z}-\frac{1}{r}\frac{\partial F_z}{\partial r}\right)\overline{k}.$$

$$\overline{F}=F_r\overline{u}_r+F_\varphi\overline{u}_\varphi+F_z\overline{u}_z;$$
$$\overline{u}_r=-\overline{i}\cos\varphi+\overline{j}\sin\varphi;$$
$$\overline{u}_\varphi=-\overline{i}\sin\varphi+\overline{j}\cos\varphi;$$

$$\bar{i} = \bar{u}_r \cos\varphi - \bar{u}_\varphi \sin\varphi;$$
$$\bar{j} = \bar{u}_r \sin\varphi + \bar{u}_\varphi \cos\varphi;$$
$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2;$$
$$dv = r\,dr\,d\varphi\,dz;$$

$$x = r\cos\varphi; \quad y = r\sin\varphi; \quad r^2 = x^2 + y^2; \ \mathrm{tg}\,\varphi = \frac{y}{x};$$

$$F_r = F_x \cos\varphi + F_y \sin\varphi;$$
$$F_\varphi = -F_x \sin\varphi + F_y \cos\varphi;$$
$$F_x = F_r \cos\varphi - F_\varphi \sin\varphi;$$
$$F_y = F_r \sin\varphi + F_\varphi \cos\varphi;$$

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}; \qquad \frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi};$$

$$\frac{\partial}{\partial r} = \frac{x}{\left(x^2+y^2\right)^{1/2}}\frac{\partial}{\partial x} \ y\frac{\partial}{\partial x} + \frac{y}{\left(x^2+y^2\right)^{1/2}}\frac{\partial}{\partial y}; \qquad \frac{\partial}{\partial\varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\bar{u}_r \times \bar{u}_\varphi = \bar{k}; \quad \bar{u}_\varphi \times \bar{k} = \bar{u}_r; \quad \bar{k} \times \bar{u}_r = \bar{u}_\varphi; \quad \bar{F}\,\bar{G} = F_r G_r + F_\varphi G_\varphi + F_z G_z;$$

$$\bar{F} \times \bar{G} = (F_\varphi G_z - F_z G_\varphi)\,\bar{u}_r + (F_z G_r - F_r G_z)\,\bar{u}_\varphi + (F_r G_\varphi - F_\varphi G_r)\,\bar{k};$$

$$\mathrm{grad}\,V = \frac{\partial V}{\partial r}\bar{u}_r + \frac{1}{r}\frac{\partial V}{\partial\varphi}\bar{u}_\varphi + \frac{\partial V}{\partial z}\bar{k}; \qquad \nabla = \bar{u}_r \frac{\partial}{\partial r} + \bar{u}_\varphi \frac{1}{r}\frac{\partial}{\partial\varphi} + \bar{k}\frac{\partial}{\partial z};$$

$$\mathrm{div}\,\bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z} = \frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z};$$

$$\nabla^2 V = \Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2};$$

$$\nabla^2 \bar{F} = \Delta\bar{F} = \left(\Delta F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\varphi}{\partial\varphi}\right)\bar{u}_r + \left(\Delta F_\varphi - \frac{F_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial F_r}{\partial\varphi}\right)\bar{u}_\varphi + \Delta F_z\,\bar{k};$$

$$\mathrm{grad}\,\mathrm{div}\,\bar{F} = \left(\frac{\partial^2 F_r}{\partial r^2} + \frac{\partial^2 F_z}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2 F_\varphi}{\partial r\partial\varphi} + \frac{1}{r}\frac{\partial F_r}{\partial r} - \frac{1}{r}\frac{\partial^2 F_\varphi}{\partial\varphi^2} - \frac{F_r}{r^2}\right)\bar{u}_r +$$

$$+\left(\frac{1}{r}\frac{\partial^2 F_z}{\partial\varphi\partial z}+\frac{1}{r^2}\frac{\partial^2 F_\varphi}{\partial\varphi^2}+\frac{1}{r}\frac{\partial^2 F_r}{\partial r\partial\varphi}+\frac{1}{r^2}\frac{\partial F_r}{\partial\varphi}\right)\bar{u}_\varphi+$$

$$\left(\frac{\partial^2 F_z}{\partial z^2}+\frac{1}{r}\frac{\partial^2 F_\varphi}{\partial\varphi\partial r}+\frac{\partial^2 F_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial F_r}{\partial z}\right)\bar{k}$$

$$\frac{\partial\bar{F}}{\partial r}=\frac{\partial F_r}{\partial r}\bar{u}_r+\frac{\partial F_\varphi}{\partial r}\bar{u}_\varphi+\frac{\partial F_z}{\partial r}\bar{k};$$

$$\frac{\partial\bar{F}}{\partial\varphi}=\bar{k}\times\bar{F}+\frac{\partial F_r}{\partial\varphi}\bar{u}_r+\frac{\partial F_\varphi}{\partial\varphi}\bar{u}_\varphi+\frac{\partial F_z}{\partial\varphi}\bar{k};$$

$$\frac{\partial\bar{F}}{\partial z}=\frac{\partial F_r}{\partial z}\bar{u}_r+\frac{\partial F_\varphi}{\partial z}\bar{u}_\varphi+\frac{\partial F_z}{\partial z}\bar{k};\qquad \frac{\partial\bar{u}_r}{\partial\varphi}=\bar{u}_\varphi;\qquad \frac{\partial\bar{u}_\varphi}{\partial\varphi}=-\bar{u}_r;$$

$$\frac{\partial\bar{u}_r}{\partial r}=\frac{\partial\bar{u}_r}{\partial z}=\frac{\partial\bar{u}_\varphi}{\partial r}=\frac{\partial\bar{u}_\varphi}{\partial z}=\frac{\partial\bar{k}}{\partial r}=\frac{\partial\bar{k}}{\partial\varphi}=\frac{\partial\bar{k}}{\partial z}=0;$$

$$\operatorname{div}\bar{u}_r=\frac{1}{r};\qquad \operatorname{div}\bar{u}_\varphi=\operatorname{div}\bar{k}=0;$$

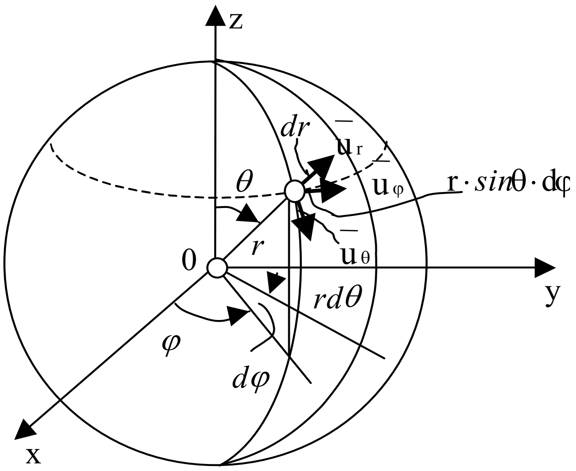
$$\operatorname{rot}\bar{u}_\varphi=\frac{\bar{k}}{r};\qquad \operatorname{rot}\bar{u}_r=\operatorname{rot}\bar{k}=0;$$

$$\operatorname{div}(\bar{u}_r\times\bar{F})=-\bar{u}_r\operatorname{rot}\bar{F};\qquad \operatorname{div}(\bar{u}_\varphi\times\bar{F})=\frac{F_z}{r}-\bar{u}_\varphi\operatorname{rot}\bar{F};$$

$$\operatorname{div}(\bar{k}\times\bar{F})=-\bar{k}\operatorname{rot}\bar{F};\quad \operatorname{rot}(\bar{u}_r\times\bar{F})=\bar{u}_r\operatorname{div}\bar{F}-\frac{\partial\bar{F}}{\partial r}-\frac{\bar{u}_r\bar{F}_r}{r}-\frac{\overline{u_z F_z}}{r};$$

$$\operatorname{rot}(\bar{u}_\varphi\times\bar{F})=-\bar{u}_r\frac{F_\varphi}{r}+\bar{u}_\varphi\operatorname{div}\bar{F}-\frac{1}{r}\frac{\partial\bar{F}}{\partial\varphi};\qquad \operatorname{rot}(\bar{k}\times\bar{F})=\bar{k}\operatorname{div}\bar{F}-\frac{\partial\bar{F}}{\partial z};$$

1.3.3 Sistemul sferic (r,θ,φ) de versori ($\bar{u}_r$, $\bar{u}_\theta$, $\bar{u}_\varphi$).



$$\bar{F} = F_r\bar{u}_r + F_\theta\bar{u}_\theta + F_\varphi\bar{u}_\varphi;$$
$$\bar{F}\,\bar{G} = F_rG_r + F_\theta G_\theta + F_\varphi G_\varphi$$
$$\bar{F}\times\bar{G} = (F_\theta G_\varphi - F_\varphi G_\theta)\,\bar{u}_r + (F_\varphi G_r - F_r G_\varphi)\,\bar{u}_\theta + (F_r G_\theta - F_\theta G_r)\,\bar{u}_\varphi;$$
$$\bar{u}_r = \bar{i}\sin\theta\cos\varphi + \bar{j}\sin\theta\sin\varphi + \bar{k}\cos\theta;$$
$$\bar{u}_\theta = \bar{i}\cos\theta\cos\varphi + \bar{j}\cos\theta\sin\varphi - \bar{k}\sin\theta;$$
$$\bar{u}_\varphi = -\bar{i}\sin\varphi + \bar{j}\cos\varphi;$$

$$\bar{i} = \bar{u}_r\sin\theta\cos\varphi + \bar{u}_\theta\cos\theta\cos\varphi - \bar{u}_\varphi\sin\varphi;$$
$$\bar{j} = \bar{u}_r\sin\theta\sin\varphi + \bar{u}_\theta\cos\theta\sin\varphi + \bar{u}_\varphi\cos\varphi;$$
$$\bar{k} = \bar{u}_r\cos\theta - \bar{u}_\theta\sin\theta;$$
$$dl^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2;$$
$$dv = r^2\,dr\sin\theta\,d\theta\,d\varphi;$$
$$x = r\sin\theta\cos\varphi;\quad y = r\sin\theta\sin\varphi;\quad z = r\cos\theta;$$
$$r^2 = x^2+y^2+z^2;\quad \mathrm{tg}\,\theta = \frac{\sqrt{x^2+y^2}}{z};\quad \mathrm{tg}\,\varphi = \frac{y}{x};$$
$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi};$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi};$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta};$$
$$\bar{u}_r\times\bar{u}_\theta = \bar{u}_\varphi;\quad \bar{u}_\varphi\times\bar{u}_r = \bar{u}_\theta;\quad \bar{u}_\theta\times\bar{u}_\varphi = \bar{u}_r;$$
$$\frac{\partial\bar{F}}{\partial r} = \frac{\partial F_r}{\partial r}\bar{u}_r + \frac{\partial F_\theta}{\partial r}\bar{u}_\theta + \frac{\partial F_\varphi}{\partial r}\bar{u}_\varphi;$$

$$\frac{\partial \bar{F}}{\partial \theta}=(\bar{u}_\varphi\times\bar{F})+\frac{\partial F_r}{\partial \theta}\bar{u}_r+\frac{\partial F_\theta}{\partial \theta}\bar{u}_\theta+\frac{\partial F_\varphi}{\partial \theta}\bar{u}_\varphi;$$

$$\frac{\partial \bar{F}}{\partial \varphi}=(\bar{k}\times\bar{F})+\frac{\partial F_r}{\partial \varphi}\bar{u}_r+\frac{\partial F_\theta}{\partial \varphi}\bar{u}_\theta+\frac{\partial F_\varphi}{\partial \varphi}\bar{u}_\varphi;$$

$$\frac{\partial \bar{u}_r}{\partial \varphi}=\sin\theta\,\bar{u}_\varphi;\quad \frac{\partial \bar{u}_r}{\partial \theta}=\bar{u}_\theta;\quad \frac{\partial \bar{u}_\theta}{\partial \theta}=\bar{u}_r;\quad \frac{\partial \bar{u}_\theta}{\partial \varphi}=\cos\theta\,\bar{u}_\varphi;$$

$$\frac{\partial \bar{u}_\varphi}{\partial \varphi}=-\bar{u}_r\sin\theta-\bar{u}_\theta\cos\theta;\qquad \frac{\partial \bar{u}_r}{\partial r}=\frac{\partial \bar{u}_\theta}{\partial r}=\frac{\partial \bar{u}_\varphi}{\partial r}=\frac{\partial \bar{u}_\varphi}{\partial \theta}=0;$$

$$\operatorname{div}\bar{u}_r=\frac{2}{r};\quad \operatorname{div}\bar{u}_\varphi=0;\quad \operatorname{div}\bar{u}_\theta=\frac{1}{r}\frac{1}{\operatorname{tg}\theta};$$

$$\operatorname{rot}\bar{u}_r=0;\quad \operatorname{rot}\bar{u}_\varphi=\frac{1}{r}\frac{\bar{u}_r}{\operatorname{tg}\theta}-\frac{1}{r}\bar{u}_\theta;\qquad \operatorname{rot}\overline{u_\theta}=\frac{1}{r}\bar{u}_\varphi;$$

$$\operatorname{div}(\bar{u}_r\times\bar{F})=-\bar{u}_r\operatorname{rot}\bar{F};$$

$$\operatorname{div}(\bar{u}_\varphi\times\bar{F})=\frac{1}{r}\frac{1}{\operatorname{tg}\theta}F_r-\frac{F_\theta}{r}-\bar{u}_\varphi\operatorname{rot}\bar{F};\quad \operatorname{div}(\overline{u_\theta}\times\bar{F})=\frac{1}{r}F_\varphi-\bar{u}_\theta\operatorname{rot}\bar{F};$$

$$\operatorname{rot}(\bar{u}_r\times\bar{F})=\bar{u}_r\operatorname{div}\bar{F}-\frac{1}{r}\bar{F}-\frac{1}{r}\bar{F}_r-\frac{\partial \bar{F}}{\partial r}$$

$$\operatorname{rot}(\bar{u}_\theta\times\bar{F})=\bar{u}_\theta\operatorname{div}\bar{F}-\frac{1}{r}\frac{1}{\operatorname{tg}\theta}(F_r\bar{u}_r+F_\theta\bar{u}_\theta)-\frac{1}{r}F_\theta\bar{u}_r-\frac{1}{r}\frac{\partial \bar{F}}{\partial \theta};$$

$$\operatorname{rot}(\bar{u}_\varphi\times\bar{F})=\bar{u}_\varphi\operatorname{div}\bar{F}-\frac{1}{r}F_\varphi\bar{u}_r-\frac{1}{r}\frac{1}{\operatorname{tg}\theta}F_\varphi\bar{u}_\theta-\frac{1}{r\,sin\theta}\frac{\partial \bar{F}}{\partial \varphi}=$$

$$=-\frac{1}{r}\frac{1}{sin\theta}\frac{\partial F_r}{\partial \varphi}\bar{u}_r-\frac{1}{r}\frac{1}{sin\theta}\frac{\partial F_\theta}{\partial \varphi}\bar{u}_\theta+\left[\frac{\partial F_r}{\partial r}+\frac{F_r}{r}+\frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right]\bar{u}_\varphi.$$

$F_r=\sin\theta\cos\varphi\,F_x+\sin\varphi\sin\theta\,F_y+\cos\theta\,F_z;$
$F_\theta=\cos\theta\cos\varphi\,F_x+\sin\varphi\cos\theta\,F_y-\sin\theta\,F_z;$
$F_\varphi=-\sin\varphi\,F_x+\cos\varphi\,F_y;$
$F_x=\sin\theta\cos\varphi\,F_r-\sin\varphi\,F_\varphi+\cos\theta\cos\varphi\,F_\theta;$
$F_y=\sin\theta\sin\varphi\,F_r+\cos\varphi\,F_\varphi+\cos\theta\sin\varphi\,F_\theta;$
$F_z=\cos\theta\,F_r-\sin\theta\,F_\theta;$

$$\frac{\partial}{\partial r}=\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}}\frac{\partial}{\partial x}+\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}}\frac{\partial}{\partial y}+\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}\frac{\partial}{\partial z};$$

$$\frac{\partial}{\partial \theta}=\frac{xz}{\left(x^2+y^2\right)^{1/2}}\frac{\partial}{\partial x}+\frac{yz}{\left(x^2+y^2\right)^{1/2}}\frac{\partial}{\partial y}-\left(x^2+y^2\right)^{1/2}\frac{\partial}{\partial z};$$

$$\frac{\partial}{\partial \varphi}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x};$$

$$\text{grad } V=\frac{\partial V}{\partial r}\bar{u}_r+\frac{1}{r}\frac{\partial V}{\partial \theta}\bar{u}_\theta+\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial V}{\partial \varphi}\bar{u}_\varphi;$$

$$\nabla=\bar{u}_r\frac{\partial}{\partial r}+\bar{u}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\bar{u}_\varphi\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi};$$

$$\text{div}\,\bar{F}=\frac{1}{r}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta)+\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}=$$

$$=\frac{\partial F_r}{\partial r}+2\frac{F_r}{r}+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{1}{r}\frac{1}{\text{tg}\theta}F_\theta+\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$\text{rot }\bar{F}=\frac{1}{r}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(F_\varphi\sin\theta\right)-\frac{\partial F_\theta}{\partial\varphi}\right]\bar{u}_r++\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{\partial}{\partial r}\left(rF_\varphi\right)\right]\bar{u}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right)-\frac{\partial F_r}{\partial\theta}\right]\bar{u}_\varphi=\left[\frac{1}{r}\frac{\partial F_\varphi}{\partial\theta}+\frac{1}{r}\frac{F_\varphi}{\text{tg}\theta}-\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial F_\theta}{\partial\varphi}\right]\bar{u}_r+$$

$$\left[\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{\partial F_\varphi}{\partial r}-\frac{F_\varphi}{r}\right]\bar{u}_\theta+\left[\frac{\partial F_\theta}{\partial r}+\frac{F_\theta}{r}-\frac{1}{r}\frac{\partial F_r}{\partial\theta}\right]\bar{u}_\varphi;$$

$$\nabla^2V=\Delta V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2V}{\partial\varphi^2}=$$

$$=\frac{\partial^2V}{\partial r^2}+2\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2V}{\partial\theta^2}+\frac{1}{r^2}\frac{1}{\text{tg}\theta}\frac{\partial V}{\partial\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2V}{\partial\varphi^2};$$

$$\nabla^2\bar{F}=\Delta\bar{F}=\left[\nabla^2F_r-\frac{2F_r}{r^2}-\frac{2\text{ctg}\theta}{r^2}F_\theta-\frac{2}{r^2}\frac{\partial F_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}\right]\bar{u}_r+$$

$$+\left[\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{F_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\varphi}{\partial\varphi}\right]\overline{u}_\theta +$$

$$+\left[\nabla^2 F_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{F_\varphi}{r^2 \sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\varphi}\right]\overline{u}_\varphi;$$

$\mathrm{rot.rot}\,\overline{F} =$

$$\left[\frac{1}{r}\frac{\partial^2 F_\theta}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial F_\theta}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2 F_r}{\partial\theta^2} + \frac{1}{r}\frac{1}{\mathrm{tg}\theta}\frac{\partial F_\theta}{\partial r} + \frac{1}{r}\frac{1}{\mathrm{tg}\theta}\frac{F_\theta}{r} - \frac{1}{r^2}\frac{1}{\mathrm{tg}\theta}\frac{\partial F_r}{\partial\theta} -\right.$$

$$\left. - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F_r}{\partial\varphi^2} - \frac{1}{r\sin\theta}\frac{\partial^2 F_\varphi}{\partial r\partial\varphi} + \frac{1}{r^2\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}\right]\overline{u}_r +$$

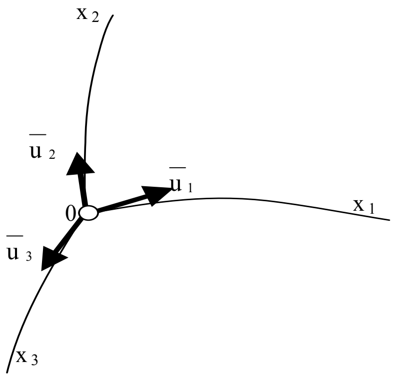
$$+\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 F_\varphi}{\partial\varphi\partial\theta} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\varphi}{\partial\varphi} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F_\theta}{\partial\varphi^2} - \frac{2}{r}\frac{\partial F_\theta}{\partial r} +\right.$$

$$\left. + \frac{1}{r}\frac{\partial^2 F_r}{\partial r\partial\theta} - \frac{\partial^2 F_\theta}{\partial r^2}\right]\overline{u}_\theta +$$

$$\left[\frac{1}{r\sin\theta}\frac{\partial^2 F_r}{\partial\varphi\partial r} - \frac{2}{r}\frac{\partial F_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 F_\varphi}{\partial\theta^2} - \frac{\partial^2 F_\varphi}{\partial r^2} - \frac{1}{r^2}\frac{1}{\mathrm{tg}\theta}\frac{\partial F_\theta}{\partial\theta} + \frac{F_\varphi}{r^2\sin^2\theta} +\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F_\theta}{\partial\theta\partial\varphi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\varphi}\right]\overline{u}_\varphi;$$

1.3.4 Sistemul curbiliniu triortogonal (x_1,x_2,x_3) de versori $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ şi h_1,h_2,h_3 coeficienți Lamé



$\overline{F} = F_1\overline{u_1} + F_2\overline{u_2} + F_3\overline{u_3}$;

$\overline{F}\,\overline{G} = F_1G_1 + F_2G_2 + F_3G_3$;

$dl_j = h_j\,dx_j$; $j=1,2,3$;

$dl^2 = h_1^2\,dx_1^2 + h_2^2\,dx_2^2 + h_3^2\,dx_3^2$;

$\mathrm{rotrot}\,\overline{F} =$

$$=\frac{1}{h_2h_3}\left\{\frac{\partial}{\partial x_2}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial x_1}h_2F_2-\frac{\partial}{\partial x_2}h_1F_1\right)\right]-\right.$$

$$\left.\frac{\partial}{\partial x_3}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial x_3}h_1F_1-\frac{\partial}{\partial x_1}(h_3F_3)\right)\right]\right\}\overline{u_1}+$$

$$+\frac{1}{h_3h_1}\left\{\frac{\partial}{\partial x_3}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial x_2}h_3F_3-\frac{\partial}{\partial x_3}h_2F_2\right)\right]-\frac{\partial}{\partial x_1}\left[\frac{h_3}{h_1h_2}\left(\frac{\partial}{\partial x_1}h_2F_2-\frac{\partial}{\partial x_2}(h_1F_1)\right)\right]\right\}\overline{u_2}$$

$$+\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial x_1}\left[\frac{h_2}{h_3h_1}\left(\frac{\partial}{\partial x_3}h_1F_1-\frac{\partial}{\partial x_1}h_3F_3\right)\right]-\frac{\partial}{\partial x_2}\left[\frac{h_1}{h_2h_3}\left(\frac{\partial}{\partial x_2}h_3F_3-\frac{\partial}{\partial x_3}(h_2F_2)\right)\right]\right\}\overline{u_3};$$

$$\text{grad } V=\frac{1}{h_1}\frac{\partial V}{\partial x_1}\overline{u_1}+\frac{1}{h_2}\frac{\partial V}{\partial x_2}\overline{u_2}+\frac{1}{h_3}\frac{\partial V}{\partial x_3}\overline{u_3};$$

$$\nabla=\overline{u_1}\frac{1}{h_1}\frac{\partial}{\partial x_1}+\overline{u_2}\frac{1}{h_2}\frac{\partial}{\partial x_2}+\overline{u_3}\frac{1}{h_3}\frac{\partial}{\partial x_3};$$

$$\text{div } \overline{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right];$$

$$\text{rot } \overline{F}=$$

$$=\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\overline{u_1}+\frac{1}{h_3h_1}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]$$

$$\overline{u_2}+\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\overline{u_3};$$

$$\nabla^2V=\Delta V=$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial V}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial V}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial V}{\partial x_3}\right)\right]$$

.

Observatii:

-sistem de coordonate cartezian:
$x_1=x,\ x_2=y,\ x_3=z,\ h_1=h_2=h_3=1.$

-sistem de coordonate cilindric:
$x_1=r,\ x_2=\varphi,\ x_3=z,\ h_1=h_3=1,\ h_2=r.$

-sistem de coordonate sferic:
$x_1=r,\ x_2=\theta,\ x_3=\varphi,\ h_1=1,\ h_2=r,\ h_3=r\sin\theta.$

-sistem de coordonate al cilindrului eliptic:

$$x_1=\xi,\ x_2=\eta,\ x_3=z,\ h_1=h_2=a\sqrt{cos^2\xi - cos^2\eta},\ h_3=1.$$

-sistem de coordonate toroidale:

$$x_1=\xi,\ x_2=\varphi,\ x_3=\psi,\ \ h_1=h_2=\frac{a}{cos\,\xi + cos\,\varphi},\ \ h_3=a\frac{\mathrm{sh}\xi}{\mathrm{ch}\xi + cos\,\varphi}$$

$$x=a\frac{cos\,\psi\,\mathrm{sh}\xi}{\mathrm{ch}\xi + cos\,\varphi},\ \ y=a\frac{sin\,\psi\,\mathrm{sh}\xi}{\mathrm{ch}\xi + cos\,\varphi},\ \ z=-a\frac{sin\,\varphi}{\mathrm{ch}\xi + cos\,\varphi}\ .$$

1.4. Aplicații.

1. Pentru orice funcție ce conține vectorul de poziție, funcțiile sale de punct se pot scrie succesiv:

$$\overline{G}_1 = k\,r^n\,\overline{r} \rightarrow \mathrm{div}\overline{G}_1 = k\,\nabla\left(r^n\overline{r}\right) = k\,\overline{r}\,\nabla\left(r^n\right) + k\,r^n\nabla\overline{r} =$$

$$= k\,\overline{r}\,n\,r^{n-1}\frac{\overline{r}}{r} + k\,r^n 3 = k\left(n+3\right)r^n$$

$$\rightarrow \mathrm{rot}\,\overline{G}_1 = \nabla\times\left(k\,r^n\,\overline{r}\right) = k\,(\nabla r^n)\times\overline{r} + k\,r^n\underbrace{\left(\nabla\times\overline{r}\right)}_{=0} =$$

$$= k\,n\,r^{n-1}\frac{\overline{r}}{r}\times\overline{r} = 0$$

$$V_1 = \frac{1}{r} \rightarrow \mathrm{grad}V_1 = \nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\,\frac{\overline{r}}{r}$$

$$V_2 = \frac{1}{r^n} \rightarrow \mathrm{grad}V_2 = \nabla\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\nabla r = \frac{-n}{r^{n+2}}\overline{r}$$

2. Se consideră funcțiile scalare f. Să se calculeze gradientul lor.

$$f(x,y,z)=\frac{z}{x^2+y^2}\rightarrow\nabla f=\frac{\partial f}{\partial r}\bar{u}_r+\frac{1}{r}\frac{\partial f}{\partial\varphi}\bar{u}_\varphi+\frac{\partial f}{\partial z}\bar{u}_z$$

sau $\nabla f=\frac{\partial f}{\partial x}\bar{i}+\frac{\partial f}{\partial y}\bar{j}+\frac{\partial f}{\partial z}\bar{k}$

$$f(r,\varphi,z)=\frac{z}{r^2}\rightarrow\nabla f=-\frac{2z}{r^3}\bar{u}_r+\frac{1}{r^2}\bar{u}_z$$

3. Se consideră funcția vectorială: $\bar{G}=x^2\,\bar{i}+e^{xy}\,\bar{j}+x\,y\,z\,\bar{k}$. Să se calculeze $\mathrm{div}\bar{G}$ şi valoarea sa în punctul P(-1,1,1).

$$\mathrm{div}\bar{G}=\nabla\bar{G}=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(e^{xy}\right)\frac{\partial}{\partial z}\left(x\,y\,z\right)=2\,x+x\,e^{xy}+x\,y$$

$$\mathrm{div}\bar{G}\Big|_{(-1,1,2)}=-2-e^{-1}-1=-3-\frac{1}{e}$$

4. Pentru funcția $\bar{G}=x^2\,y^2\,\bar{i}+2\,x\,y\,z\,\bar{j}+z^2\bar{k}$, să se calculeze $\mathrm{rot}\bar{G}$ şi valoarea sa în punctual $P_0(1,-2,1)$.

$$\mathrm{rot}\bar{G}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ G_x & G_y & G_z\end{vmatrix}=-2\,x\,y\,\bar{i}+\left(\,2\,y\,z-2\,x^2\,y\,\right)\,\bar{k}\ ;\quad \mathrm{rot}\bar{G}\Big|_{P_c}=4\bar{i}$$

5. Să se calculeze circulația vectorului $\bar{G}=x^2\bar{i}+y^2\bar{j}+z^2\bar{k}$ pe dreapta P_1P_2 sau pe curba $P_1P_2P_3P_4$ din figura 1.4.

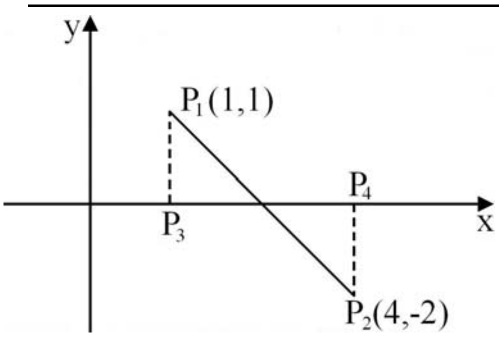


Fig. 1.4

Ecuația dreptei P_1P_2 este $y=-x+2$, iar elementul de linie este: $\overline{d\ell}=dx\,\bar{i}+dy\,\bar{j}+dz\,\bar{k}$. Deci

$$I=\int_{P_2}^{P_1}\bar{G}\,\overline{d\ell}=\int_1^4\left[x^2\bar{i}+\left(-x+2\right)^2\bar{j}\right]\left(\bar{i}-\bar{j}\right)\,dx=\left(2x^2-4x\right)\Big|_1^4=18$$

Circulația pe conturul $P_1-P_3-P_4-P_2$ este:

$$I' = \int_{P_1}^{P_3} \overline{G}\,\overline{dl} + \int_{P_3}^{P_4} \overline{G}\,\overline{dl} + \int_{P_4}^{P_2} \overline{G}\,\overline{dl} = \int_{P_1}^{P_2} \overline{G}\left(-\overline{j}\,dy\right) + \int_{P_3}^{P_4} \overline{G}\left(\overline{i}\,dx\right) + \int_{P_4}^{P_2} \overline{G}\left(-\overline{j}\,dy\right) =$$

$$= \int_0^1 -y^2\,dy + \int_1^4 x^2\,dx + \int_{-2}^0 -y^2\,dy = -\frac{1}{3} + \frac{63}{3} + \frac{8}{3} = 18$$

deci valoarea integralei între punctele P_1 - P_2 nu depinde de drumul de integrare. Astfel de câmpuri poartă numele de câmpuri potenţiale şi derivă dintr-o funcţie scalară prin intermediul gradientului.

6. Să se verifice teorema lui Stokes cu vectorul $\overline{A} = y\,\overline{i} - x\,\overline{j}$ pentru curba Γ din figura 1.5.

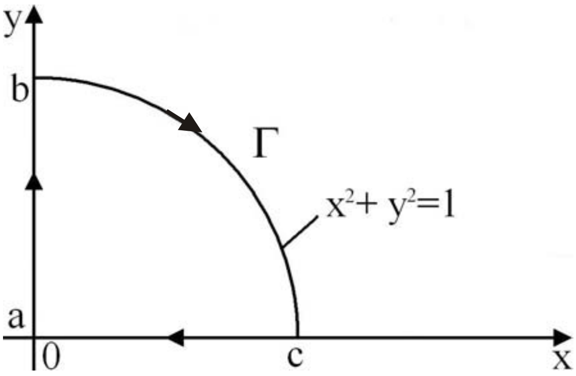


Fig. 1.5

$\overline{A}\,\overline{d\ell} = y\,dx - x\,dy$, deoarece

$\overline{d\ell} = dx\,\overline{i} + dy\,\overline{j}$

Pe porţiunea (a-b), având

$$\begin{cases} x = 0 \\ dx = 0 \end{cases} \rightarrow \overline{A}\,\overline{d\ell} = 0$$

Pe porţiunea (b-c):

$$y = \sqrt{1 - x^2} \rightarrow$$

$$\rightarrow dy = -\frac{x\,dx}{y} = \frac{-x\,dx}{\sqrt{1 - x^2}}$$

$$\rightarrow \overline{A d\ell} = \sqrt{1 - x^2}\,dx + \frac{x^2\,dx}{\sqrt{1 - x^2}} = \frac{dx}{\sqrt{1 - x^2}}$$

$$\int_b^c \overline{A d\ell} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2}$$

Pe porţiunea (c-a): $\begin{cases} y = 0 \\ dy = 0 \end{cases} \rightarrow \overline{A d\ell} = 0$

Atunci: $\oint_\Gamma \overline{A}\,\overline{d\ell} = \int_a^b \overline{A}\,\overline{d\ell} + \int_b^c \overline{A}\,\overline{d\ell} + \int_c^a \overline{A}\,\overline{d\ell} = 0 + \frac{\pi}{2} + 0 = \frac{\pi}{2}$

De asemenea: $\mathrm{rot}\overline{A} = \nabla \times \overline{A} = \nabla \times \left(y\,\overline{i} - x\,\overline{j}\right) = -2\overline{k}$ şi $\overline{ds} = -dx\,dy\,\overline{k}$

Rezultă: $\int_{S_\Gamma} \mathrm{rot}\overline{A}\,\overline{ds} = \int_{x=0}^{x=1} dx \int_{y=0}^{\sqrt{1-x^2}} 2\,dy = 2\,S_\Gamma = 2\,\frac{\pi}{4} = \frac{\pi}{2}$

Deci: $\oint_\Gamma \overline{A}\,\overline{d\ell} = \int_{S_\Gamma} \mathrm{rot}\overline{A}\,\overline{ds} = \frac{\pi}{2}$,

Ceea ce înseamnă că se verifică teorema lui Stokes.